presented in Table III. They give a revised contour diagram of the polarization as a function of the energy and the scattering angle, in the energy range 3.7 to 4.7 MeV, as shown in Fig. 6.

It appears from the foregoing analysis that polarization measurements provide a very sensitive method of determining the complete and unique phase-shift prediction, provided that precise cross-section angular distributions are available at the same proton energies where the polarization has been measured.

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Dynamic Calculations of Fission of an Axially Symmetric Liquid Drop

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A dynamical treatment for an axially symmetric liquid drop is presented. A general parametrization of the nuclear shape is introduced. The framework is suitable for the inclusion of a nuclear-energy term in the Hamiltonian. A particular mode of fission is solved numerically, to obtain saddle-point and scission-point shapes, and kinetic-energy distributions.

I. INTRODUCTION

THE fission of the atomic nucleus into two or more parts is a phenomenon of well-established importance. The accumulation of experimental data on the various aspects of fission is constantly increasing. From a theoretical point of view, however, our present understanding of it dates back practically to the late thirties when Bohr and Wheeler proposed the liquid-drop model of fission.[1] This classical model is essentially the only model that has been dealt with. But even within its own frame of reference, numerical calculations have been scarce and unsystematic. They were mostly of a static nature[2] (saddle-point shapes, etc.), as opposed to the more intricate problems of the statistic-mechanical[3,4] and dynamic[5,6] aspects of fission, treated separately.

Perhaps the most outstanding feature of fission is its asymmetry. In spontaneous fission (and fission produced by low-energy projectiles) nuclei break mostly into two unequal parts. The classical liquid-drop model completely fails to explain this effect. However uncertain be its other quantitative implications, it unambiguously predicts the fission to be symmetric. A qualitative explanation is proposed by taking into account the shell structure of the nucleus. The final products of fission tend to abound around mass numbers that represent strongly bound almost-magic nuclei. The liquid-drop model and the shell model are, however, based on completely different basic assumptions. The first is a strongly interacting model of the nucleons, whereas the second is essentially an independent-particle model. More than a mere reconciliation between these two extremes is needed in order to be able to treat quantitatively the effects of nuclear forces on fission. Moreover, the shell structure is a characteristic of a spherical, nonexcited nucleus, while the fissioning process involves excitations and large distortions of the nuclear shape.

The purpose of this work is to treat fission as a dynamical process, and to incorporate later nuclear structure effects into this treatment.

The formalism used is a classical one. It is only through the determination of initial conditions that quantum effects affect the problem. The nucleus is assumed to be axially symmetric. This assumption is not as restrictive as it might first appear, since we are not interested in minor details of structure or distributions but rather in gross average properties and their dependence on nuclear characteristics. This assumption amounts to an over-all averaging of the fission process. Thus, strong local distortions of the nuclear surface are practically not considered.

The effects of the nuclear forces in producing asymmetric (mainly pear-shaped) nuclei have already been studied to some extent.[7,8] Without dealing here with the exact nature of these calculations, we would like to stress that they treat only the equilibrium state of the nucleus.[9] They do not affect the saddle-point shape, the scission point, or the evolution of the system between these points. In fission, however, it is these stages that

[1] N. Bohr and J. A. Wheeler, Phys. Rev. **56**, 426 (1939).
[2] For example: S. Cohen and W. J. Swiatecki, Lawrence Radiation Laboratory Report UCRL-10450, 1962 (unpublished).
[3] P. Fong, Phys. Rev. **122**, 1543 (1961).
[4] T. D. Newton, in Proceedings of the Symposium on the Physics of Fission, Chalk River, Ontario, 1956 (unpublished).
[5] D. L. Hill and J. A. Wheller, Phys. Rev. **89**, 1102 (1953).
[6] D. L. Hill, in *Proceedings of the Second United Nations International Conference on the Peaceful uses of Atomic Energy, Geneva, 1958* (United Nations, Geneva, 1958).
[7] H. Faissner and K. Wildermuch, Nucl. Phys. (to be published).
[8] K. Lee and D. R. Inglis, Phys. Rev. **108**, 774 (1957).
[9] I. Dutt and P. Mukherjee, Phys. Rev. **124**, 888 (1961).

contain most of the physical information.[10] The main idea is, therefore, to take into account the nuclear effects not only at a particular stage but along the whole fission process.

The basic assumptions and the general procedure followed in performing such calculations are given in Sec. II. In Sec. III, a practical method for doing a large variety of calculations of this nature is introduced. Sec. IV deals with the computation of the Coulomb energy and its derivatives, which has to be performed with a high degree of accuracy. In the last sections (V and VI) a special case is quantitatively solved. However, since the main objective of these calculations was to provide a numerical check for the methods outlined, the quoted results are of a fragmentary nature.

II. THE EQUATIONS OF MOTION

The assumption which underlies the following treatment is that the Hamiltonian of the three-dimensional liquid drop is uniquely determined by its two-dimensional boundary. In the case where the drop has axial symmetry, it is further assumed that the physical information is completely contained in the generating line shape. This is certainly true for static properties of the system. For dynamical properties (especially the kinetic energy), explicit relations have to be assumed or established in order to justify this approach. For the case of an irrotational flow of an incompressible, nonviscous fluid, the motion of the surface determines uniquely the motion of the volume within. The exact determination, however, is usually very complicated. Simplifying assumptions on the nature of the motion are therefore made. The explicit assumption used here is one suggested by Wheeler.[11] It describes the flow as a flow of circular layers of fluid. That is, all points which are at one time on a plane perpendicular to the symmetry axis will continue to be on such a plane ever after. A disk of fluid, therefore, will change its thickness and radius with time, but will still remain a disk.

We assume, therefore, an axially symmetric system, bounded by a curve $y(x)$ (y and x are the first two cylindrical coordinates). $y(x)$ is a function of time, and its behavior is subjected to the classical laws of motion in the Hamiltonian or Lagrangian formalism.

The Hamiltonian H of the system is of the general form

$$H=H(y(x),\dot{y}(x)), \tag{1}$$

where the dependence of H on the derivatives of $y(x)$ with respect to x is implicitly absorbed into the dependence on $y(x)$. Defining the canonical coordinate with respect to $y(x)$

$$p(x)=\delta H/\delta\dot{y}(x). \tag{2}$$

The equations of motion are

$$\dot{y}(x)=\partial H(y(x),p(x))/\partial p(x), \tag{3}$$

$$\dot{p}(x)=-\partial H(y(x),p(x))/\partial y(x). \tag{4}$$

Expressed in this way, x is describable as a parameter specifying an infinite set of independent coordinates $y(x)$. The transformation to canonical coordinates is performed making use of Wheeler's assumption about the nature of the motion. The potential energy depends only on $y(x)$. The kinetic energy can be expressed in the form

$$T=\frac{1}{2}\int\int G(x,x')\dot{y}(x)\dot{y}(x')dxdx', \tag{5}$$

where (see Appendix A) the kernel G is given by

$$G(x,x')$$

$$=\pi\sigma\left\{y(x')\frac{\partial y}{\partial x}(x)\theta(x,x')+y(x)\frac{\partial y}{\partial x}(x')\theta(x',x)+y(x)y(x')\right.$$

$$\left.\left[\tfrac{1}{2}\delta(x,x')+\int\frac{1+\frac{1}{2}\left[\frac{\partial y}{\partial x}(x'')\right]^2}{y(x'')^2}\theta(x'',x,x')dx''\right]\right\}, \tag{6}$$

where

$$\theta(\alpha_0,\alpha_1,\cdots,\alpha_n)=1, \quad \text{if} \quad \prod_{i=1}^{n}(\alpha_i-\alpha_0)>0,$$

$$=0, \quad \text{if} \quad \prod_{i=1}^{n}(\alpha_i-\alpha_0)<0.$$

Using the definition of Eq. (2), we have

$$p(x)=\int G(x,x')\dot{y}(x')dx'. \tag{7}$$

Defining the inverse kernel $G^{-1}(x,x')$ by the equation

$$\int G^{-1}(x,x')G(x',x'')dx'=\delta(x,x''), \tag{8}$$

the kinetic energy can be expressed in terms of $p(x)$ as

$$T=\frac{1}{2}\int\int G^{-1}(x,x')p(x)p(x')dxdx'. \tag{9}$$

The Hamilton equation of motion for $\dot{y}(x)$, Eq. (3), is thus simply given by

$$\dot{y}(x)=\int G^{-1}(x,x')p(x')dx'. \tag{10}$$

[It should be noted that the integrations are extended only over the region where $y(x)$ does not vanish.] The

[11] J. A. Wheeler (unpublished).

[10] J. A. Wheeler, in *Fast Neutron Physics*, edited by J. B. Marion and J. L. Fowler (Interscience Publishers, Inc., New York, 1962), Part III.

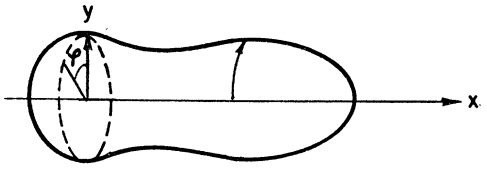

FIG. 1. Cylindrical coordinates used to describe the axially symmetric liquid drop.

condition of incompressibility of the liquid is equivalent to the condition of constant volume. The volume

$$V=\pi\int y^2dx \tag{11}$$

should, therefore, be a constant of motion, or

$$dv/dt\approx\int\dot{y}ydx\equiv 0. \tag{12}$$

In the dynamical treatment $\dot{y}(x)$ depends solely on T through Hamilton's equation. The incorporation of Wheeler's assumption into the kinetic energy assures the conservation of volume, namely,

$$\int\int y(x)G^{-1}(x,x')p(x')dxdx'\equiv 0. \tag{13}$$

III. PARAMETRIC DESCRIPTION OF THE MOTION

The general integration of the equations of motion for the axially symmetric liquid drop is computationally quite involved (Fig. 1). It is therefore practical to limit the form of the generating line to a family of curves, parameterized by a set of parameters $\alpha_1, \cdots, \alpha_n$. The laws of conservation of volume and of linear momentum have to be incorporated *a priori* into the equations, by properly choosing the family of curves. Instead of a functional variable $y(x)$, we shall have a finite set of parameters for our dynamic variables. (An example for such a treatment is the work of Nix,[12] where the liquid drop is assumed to have the form of two spheroids, overlapping at first and separated at a later stage.)

We thus have

$$y=y(\alpha_1,\cdots,\alpha_n:x), \tag{14}$$

$$\dot{y}=\sum\frac{\partial y}{\partial\alpha_i}\dot{\alpha}_i, \tag{15}$$

$$\frac{Dy}{Dt}=\dot{y}+\frac{\partial y}{\partial x}\frac{Dx}{Dt}. \tag{16}$$

Substituting (15) into (5) and (6), we immediately obtain the kinetic energy as a function of the derivatives of the parameters

$$T=\tfrac{1}{2}\sum_{i,j}G_{ij}\dot{\alpha}_i\dot{\alpha}_j, \tag{17}$$

where

$$G_{ij}=\int\int G(x,x')\frac{\partial y}{\partial\alpha_i}(x)\frac{\partial y}{\partial\alpha_j}(x')dxdx'. \tag{18}$$

The conjugate momenta $\beta_1, \cdots, \beta_n$ are similarly defined as

$$\beta_i\equiv\partial T/\partial\dot{\alpha}_i=\sum_j G_{ij}\dot{\alpha}_j. \tag{19}$$

With the inverse matrix $(G^{-1})_{ij}$, we express

$$T=\tfrac{1}{2}\sum_{i,j}(G^{-1})_{ij}\beta_i\beta_j \tag{20}$$

and Hamilton's equation for $\dot{\alpha}$ is simply

$$\dot{\alpha}_i=\sum_j(G^{-1})_{ij}\beta_j. \tag{21}$$

The matrix of the kinetic energy can be evaluated directly, without referring to the kernel $G(x,x')$, in a way which is very convenient for computations. Let us expand

$$\frac{Dx}{Dt}(x)=\sum_i A_i(x)\dot{\alpha}_i, \tag{22}$$

$$\frac{Dy}{Dt}(x)=\sum_i B_i(x)\dot{\alpha}_i. \tag{23}$$

The functions $A_i(x)$ and $B_i(x)$ obey the local relation

$$B_i(x)=A_i(x)\frac{\partial y}{\partial x}(x)+\frac{\partial y}{\partial\alpha_i}(x). \tag{24}$$

In terms of these functions, we immediately obtain

$$G_{ij}=\pi\sigma\int y(x)^2\{A_i(x)A_j(x)+\tfrac{1}{2}B_i(x)B_j(x)\}dx, \tag{25}$$

where, here as well, the integration is extended only over the region for which $y(x)\neq 0$. If we know for each such simply connected region its extremal point x_m as a function of the parameters $\{\alpha_i\}$, we can use it in Eq. (A8) by integrating up to x_m. In this case we get

$$A_i(x)=\frac{2}{y(x)^2}\int_x^{x_m}y(x')\frac{\partial y}{\partial\alpha_i}(x')dx'+\frac{\partial x_m}{\partial\alpha_i} \tag{26}$$

or

$$A_i(x)=\frac{\partial x_m}{\partial\alpha_i}+\frac{1}{y(x)^2}\frac{\partial}{\partial\alpha_i}\int_x^{x_m}y(x')^2dx'.$$

In this way the parametric description is easily handled in practical calculations.

[12] J. R. Nix, Lawrence Radiation Laboratory Report UCRL-10695, 1963 (unpublished).

IV. THE COULOMB ENERGY

The Coulomb energy of any uniformly charged body with density ρ is given by a six-dimensional integral

$$E_c=\rho^2\int\int\frac{d\tau_1 d\tau_2}{r_{12}}. \tag{27}$$

In the case of an axially symmetric body, we can simplify the computational procedure by considering the electrostatic interaction between circular rings of charge. The interaction energy between two coaxial, infinitely thin rings of radii r_1 and r_2 a distance R apart and of unit-length density is given by

$$V_{RR}(1,2)=\frac{4r_1r_2}{[(r_1+r_2)^2+R^2]^{1/2}}\times E\left(\left[\frac{4r_1r_2}{(r_1+r_2)^2+R^2}\right]^{1/2}\right), \tag{28}$$

where $E(\theta)$ is the complete elliptic integral of the first kind. E_c can then be calculated by a double surface integral, performed over the area enclosed by the x axis and the line $y(x)$

$$E_c=\rho^2\int\int V_{RR}(1,2)ds_1ds_2. \tag{29}$$

A different computational approach[13] is to sum over interactions between thin cylinders of charge. In the Hamiltonian equations of motion, however, the Coulomb energy does not appear explicitly, but only in its derivative. The variation of the electrostatic energy as a result of a variation $\delta y(x)$ in the shape of the generating line is simply

$$\delta E_c=2\pi\rho\int\delta y(x)y(x)V(y(x),x)dx, \tag{30}$$

where $V(y(x),x)$ is the electrostatic potential on the surface of the charge at the coordinate x resulting from the charge distribution defined by $y(x)$. Equation (30) implies that we do not have to calculate the self-energy of the liquid drop itself at each step in order to find its derivative. Rather,

$$\frac{\delta E_c}{\delta y(x)}=2\pi\rho y(x)V(y(x),x) \tag{31}$$

and can be evaluated directly. Thus expressed, δE_c is a four-dimensional integral, whereas E_c itself is a six-dimensional integral. This renders the use of numerical differentiation of E_c unnecessary, and extracts a grave source of inaccuracy from the calculation.

In the parametric description approach, the derivatives of E_c are obtained by replacing the variation δy by its expansion in $\delta\alpha_i$'s. Thus

$$\frac{\partial E_c}{\alpha_i}=2\pi\rho\int\frac{\partial y}{\partial\alpha_i}(x)y(x)V(y(x),x)dx. \tag{32}$$

V. A SPECIAL CASE

We shall here present a quantitative example[14] of a dynamical treatment of the liquid drop, using three parameters as dynamical variables. The integration of the equations of motion for this choice is simplified by the fact that most of the terms in the Hamiltonian and their derivatives can be expressed in closed form. This circumstance enables us to check the accuracy of the computational procedure for general cases, where such a simplification does not occur.

The liquid drop is assumed to have the form of two identical spheres with a canonical quadratic surface (i.e., an ellipsoid or a hyperboloid) between them. This family of curves contains pre-scission as well as post-scission forms. A particular case which occurs too is a special tripartition of the liquid drop. All these possibilities are illustrated in Fig. 2. Three parameters are necessary to specify the function $y(x)$. Two of these are

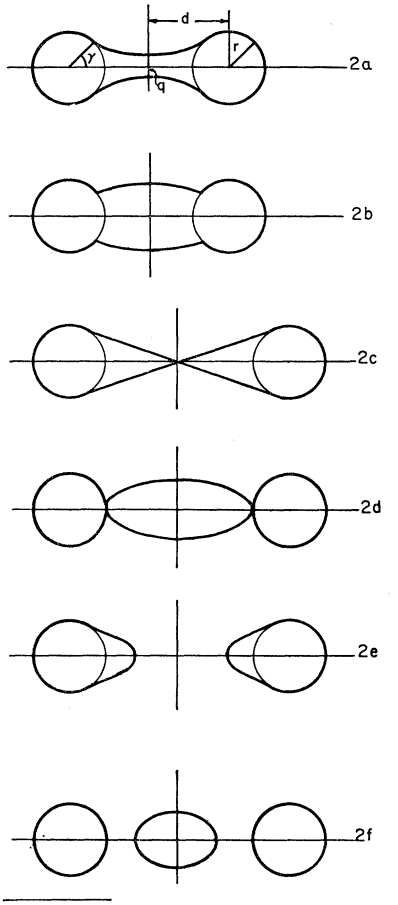

FIG. 2. All possible shapes in the family of curves with the partiuclar choice of parameters. q, the thickness of the neck, and γ (or its cosine) can be used alternately. Cuves 2a, 2c, and 2e give pre-scission, scission and post-scission typical stages of binary fission. 2b, 2d, and 2f are the corresponding curves for ternary fission.

[13] R. Beringer, Phys. Rev. **131**, 1402 (1963).

[14] This was suggested by Dr. W. J. Swiatecki, to whom the author is very much indebted.

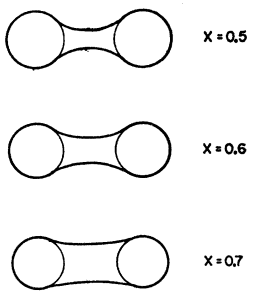

FIG. 3. Saddle-point shapes for region of fissionability parameter 0.5–0.7, where they reproduce fairly well shapes obtained by use of a greater number of distortion parameters.

the radius of the spheres r and the distance $2d$ between their centers. One additional parameter describes the form of the neck. This may be taken as the angular coordinate of the point on the sphere where the neck joins it, or as the width of the neck at its midpoint. The conservation of volume then serves to determine uniquely the analytic form of the curve $y(x)$. When the equations of motions are integrated, the reduction to a set of coordinates which are independent for a prescribed volume, assures the conservation of volume automatically.

As a result of the quadratic nature of $y(x)$, the surface energy is a simple expression. This obviously applies also to the Coulomb self-energy of the spheres, and to their interaction energy with the neck. Only the Coulomb self-energy of the neck necessitates numerical evaluation.

This family of curves describes only symmetric fission though it can easily be generalized to asymmetric fission as well. The whole description, however, mainly applies for lower values of the fission parameter X, where it fairly reproduces saddle-point shapes calculated using a large number of distortion parameters.[2] For low X values, symmetric fission is indeed relatively more abundant.

One of the major points in going through a dynamical calculation is to find the way in which the large excess of potential energy is released during fission. The difference in potential energy between a sphere of given charge and volume, and a system of two infinitely separated spheres of half that charge, is of the order of magnitude of 200 MeV. Most of this energy is converted into the kinetic energy of the fission fragments, the residue being the excitation energy of these fragments. The exact division of the energy is determined primarily during the first stages of fission, i.e., between saddle point and scission point. Later there is practically no interchange of energy between the two modes, and the system can be represented by two point charges being electrostatically repelled by one another. For this reason the integration of the equation of motion was not carried far beyond the scission point. Moreover, the restriction on the form of the fragments induces at that stage a transformation of excitation energy into kinetic energy which is a result of an artificial smoothing of the nuclear distortion.

The equations of motion—a set of six coupled linear differential equations—were integrated by the Runge-Kutta method, using a variable time increment. The results of the calculations and their analysis are presented in the next section.

VI. RESULTS

The calculations were performed in two steps. In the first step, saddle-point shapes were found for different values of the fissionability parameter X. In the second step, the equations of motion were integrated starting at saddle-point with various initial conditions on the first derivatives of the coordinates. This amounts to giving the system energy above the threshold energy, and dividing it between the various degrees of freedom. However, no weighting of the different choices of initial conditions was made. A special emphasis was laid on investigating the characteristics of the neck during the process.

Figure 3 gives saddle point shapes for $X=0.5$, 0.6, 0.7. The thickening of the neck at the expense of the sphere leads to cylinder-like saddle-point shapes at higher values of X. To compare these results, with those of Ref. 2, ξ, the deformation energy in units of the sphere's surface energy is calculated, yielding $\xi=0.0962$ for $X=0.5$, $\xi=0.0583$ for $X=0.6$, and $\xi=0.0246$ for $X=0.7$. The valve $Z=90$ is chosen throughout the calculations.

Figure 4 illustrates the dependence of the neck thickness on time. Three types of such dependence are shown. In (i), starting at rest, a slight thickening of the neck at first is followed by a rather abrupt approach to the scission point. (ii) shows typical oscillations of the thickness as it decreases, obtained for medium velocities $\dot{r}$ and $\dot{d}$. (iii) is a typical curve obtained when all the energy excess at the saddle point is given as positive $\dot{r}$, and the scission is less abrupt. The energy for (ii) and (iii) is approximately 2.5 MeV above threshold. We should note that the time between saddle point and scission point is between 10^{-22} and 10^{-23} sec, and depends largely on the initial conditions at the saddle point. This dependence is summarized in Table I and the dependence on X for similar initial condition is shown in Table II. The knowledge of this time is important for determining whether the process can be viewed as

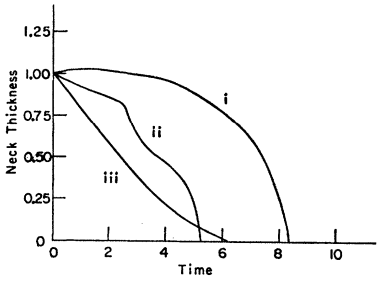

FIG. 4. Neck thickness variation with time between saddle point and scission point, for three different typical initial conditions at saddle point. The thickness is measured in units of the original neck thickness, and time in natural units ($m_p \equiv 1$; $r_p = 1.2\times10^{-13}$ cm $\equiv 1$; $e_p \equiv 1$; 1 time unit $\equiv 1.12 \times 10^{-22}$ sec).

TABLE I. The saddle-point to scission-point time as a function of initial conditions at saddle point. The units used are the natural ones; the charge, mass, and radius of the proton are taken to be unity; the time unit is then 1.12×10^{-22} sec. $X=0.70$.

p_r \ p_d	0	4.12	5.83	7.16	8.28
−8.0	8.92	8.07	7.05	5.99	5.01
−4.0	8.47	7.48	6.43	5.39	4.35
0	8.33	7.30	6.18	5.09	3.97
4.0	8.02	6.94	5.52	4.21	2.91
8.0	7.61	6.23	4.71	3.19	1.63

TABLE II. The saddle-point to scission-point time as a function of X, in natural units: (i) For $p_d=p_r=0$, (ii) For $p_d=8.28$, $p_r=8.0$.

$X=$	0.5	0.55	0.6	0.65	0.7
(i)	6.41	7.09	7.57	7.97	8.33
(ii)	1.10	1.22	1.35	1.49	1.63

adiabatic or not. This determination plays a central role in calculations of emission of neutrons from a snapping neck.[15]

The fraction of the mass in the neck at the moment of scission is also of interest. Its dependence on initial conditions at saddle point is shown in Table III.

Table IV gives the dependence of the final kinetic energy of the fragment on saddle-point conditions. The initial value of P_q (the momentum associated with the neck thickness) was taken to be zero for all cases, except for those where $P_d=0$ and $P_r<0$. For these cases P_q was assigned a minimal value to allow the system to fission.

TABLE III. The fraction of mass in the neck, at scission time, as a function of initial conditions at saddle point. $X=0.70$.

p_r \ p_d	0	4.12	5.83	7.16	8.28
−8.0	0.144	0.156	0.170	0.191	0.214
−4.0	0.132	0.140	0.149	0.158	0.165
0	0.121	0.125	0.132	0.141	1.151
4.0	0.110	0.112	0.117	0.127	0.143
8.0	0.097	0.102	0.108	0.115	0.124

TABLE IV. The total kinetic energy of both fragments as a function of initial conditions at saddle point. The unit of energy is the difference between the potential energy of one charged sphere and two infinitely separated spheres of half size. The quoted values of initial p_d correspond to increases of the initial kinetic energy by 0.04 of that unit. $X=0.70$.

p_r \ p_d	0	4.12	5.83	7.16	8.28
−8.0	0.898	0.937	0.975	1.011	1.042
−4.0	0.912	0.951	0.990	1.026	1.059
0	0.926	0.964	1.001	1.036	1.070
4.0	0.939	0.979	1.018	1.057	1.095
8.0	0.951	0.992	1.034	1.075	1.115

[15] R. W. Fuller, Phys. Rev. **126**, 684 (1962).

The expected negative correlation between high kinetic energies and massive necks is indeed observed. A larger neck corresponds to a large distortion of the fragment associated with high excitation energy.

APPENDIX A

We shall here express the kinetic energy of the liquid drop in terms of $\dot{y}(x)$, using Wheeler's assumption on the nature of the motion.

The irrotationality of the motion implies that there is no tangential component of the velocity in the y–z plane. The kinetic energy of a thin disk of fluid is thus uniquely determined by the rate of change of its position along the x axis and its radius (or thickness). Performing the integration over such a disk, we obtain

$$\Delta T=\tfrac{1}{2}\Delta m\left[\left(\frac{Dx}{Dt}\right)^2+\frac{1}{2}\left(\frac{Dy}{Dt}\right)^2\right], \tag{A1}$$

where the operator D/Dt gives the time derivative with respect to motion of the fluid, i.e., it gives the velocity components of a particle of fluid found at (x,y) at a certain time. Integrating over dm and assuming uniform distribution with density σ we get

$$T=\tfrac{1}{2}\pi\sigma\int y^2\left[\left(\frac{Dx}{Dt}\right)^2+\frac{1}{2}\left(\frac{Dy}{Dt}\right)^2\right]dx \tag{A2}$$

y, Dx/Dt, and Dy/Dt are all functions of x. $\dot{y}(x)$, which is the partial derivative of y with respect to time, $\partial y(x)/\partial t$, is related to Dy/Dt through

$$\frac{Dy}{Dt}=\dot{y}+\frac{\partial y}{\partial x}\frac{Dx}{Dt}. \tag{A3}$$

Therefore,

$$T=\tfrac{1}{2}\pi\sigma\int y^2\left\{\tfrac{1}{2}\dot{y}^2+\frac{\partial y}{\partial x}\frac{Dx}{Dt}\dot{y}+\left[1+\frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2\right]\left(\frac{Dx}{Dt}\right)^2\right\}dx$$
$$\equiv T_0+T_1+T_2, \tag{A4}$$

where T was split into terms depending on different powers of Dx/Dt. To evaluate Dx/Dt, we define

$$V^{(+)}(x)=\pi\int_x y(x')^2dx', \tag{A5}$$

$$V^{(-)}(x)=\pi\int^x y(x')^2dx'. \tag{A6}$$

The Wheeler condition implies that the order of points of the fluid along the x axis is preserved, or

$$\frac{D}{Dt}V^{(+)}(x)=\frac{D}{Dt}V^{(-)}(x)\equiv 0. \tag{A7}$$

Thus

$$y(x)^2\frac{Dx}{Dt}(x)=2\int_x y(x')\dot{y}(x')dx', \tag{A8}$$

$$y(x)^2\frac{Dx}{Dt}(x)=-2\int^x y(x')\dot{y}(x')dx'. \tag{A9}$$

Inserting (A8) in T_1 we get

$$T_1=\pi\sigma\int dx\frac{\partial y}{\partial x}(x)\dot{y}(x)\int^x y(x')\dot{y}(x')dx'$$

$$=\pi\sigma\int\int dxdx'\frac{\partial y}{\partial x}(x)y(x')\dot{y}(x)\dot{y}(x')\theta(x,x'), \tag{A10}$$

where

$$\theta(x,x')=\begin{cases}1 & \text{if} \quad x<x'\\ 0 & \text{if} \quad x>x'\end{cases}.$$

Equation (A10) can be written in a symmetric way

$$T_1=\tfrac{1}{2}\pi\sigma\int\int dxdx'\dot{y}(x)\dot{y}(x')$$

$$\times\left\{\frac{\partial y}{\partial x}(x)y(x')\theta(x,x')+\frac{\partial y}{\partial x}(x')y(x)\theta(x'x)\right\} \tag{A10a}$$

or

$$T_1=\frac{1}{2}\int\int G_1(x,x')\dot{y}(x)\dot{y}(x')dxdx', \tag{A11}$$

with

$$G_1(x,x')=\pi\sigma\left\{y(x')\frac{\partial y}{\partial x}(x)\theta(x,x')+y(x)\frac{\partial y}{\partial x}(x')\theta(x',x)\right\}. \tag{A11a}$$

To evaluate the quadratic term T_2 we shall square Eqs. (A7) and (A8), obtaining

$$y(x)^2\left\{\frac{Dx}{Dt}(x)\right\}^2$$

$$=\frac{4}{y(x)^2}\int_x\int_x y(x')y(x'')\dot{y}(x')\dot{y}(x'')dx'dx'', \tag{A12}$$

$$y(x)^2\left\{\frac{Dx}{Dt}(x)\right\}^2$$

$$=\frac{4}{y(x)^2}\int^x\int^x y(x')y(x'')\dot{y}(x')\dot{y}(x'')dx'dx''. \tag{A13}$$

Combining these two expressions we can write

$$y(x)^2\left\{\frac{Dx}{Dt}(x)\right\}^2=\frac{2}{y(x)^2}\int\int y(x')y(x'')\dot{y}(x')\dot{y}(x'')\times\theta(x,x',x'')dx'dx'', \tag{A14}$$

where

$$\theta(x,x',x'')=1, \quad \text{if} \quad (x'-x)(x''-x)>0,$$
$$=0, \quad \text{if} \quad (x'-x)(x''-x)<0.$$

Here, as for T_1, we get, by substituting,

$$T_2=\frac{1}{2}\int\int G_2(x,x')\dot{y}(x)\dot{y}(x')dxdx' \tag{A15}$$

with

$$G_2(x,x')=\pi\sigma y(x)y(x')\times\int\frac{1+\frac{1}{2}\left(\frac{\partial y}{\partial x}(x'')\right)^2}{y(x'')^2}\theta(x'',x,x')dx''. \tag{A15a}$$

T_0 can be written in a similar way as

$$T_0=\frac{1}{2}\int\int G_0(x,x')\dot{y}(x)\dot{y}(x')dxdx', \tag{A16}$$

where

$$G_0(x,x')=\tfrac{1}{2}\pi\sigma y(x)y(x')\delta(x,x'). \tag{A16a}$$

Combining now (A16), (A15), and (A11), we obtain Eq. (5) with

$$G=G_0+G_1+G_2. \tag{A17}$$

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